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It is shown that the L_2 norm of sufficiently smooth solutions of initial-history value problems associated with the nonlinear, one-dimenstional model of viscoelastic response given by

$u_{tt} = a(0)\sigma(u_x)_x - \int_0^t a_\tau(t-\tau)\sigma(u_x)_x d\tau + \mathcal{F}$, must grow quadratically in time

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) on any interval $[0,T)$ where the H^1 norm of u is bounded from above, provided

(i) $\alpha\Sigma(\zeta) \geq \zeta\Sigma'(\zeta)$, $\forall\zeta \in R^1$, some $\alpha > 0$ where $\Sigma(\zeta) = \int_0^\zeta \sigma(e)de$

(ii) $\exists\bar{\sigma} > 0$ such that $|\sigma'(\zeta)| \leq \bar{\sigma}$, $\forall\zeta \in R^1$

No sign definiteness restrictions are imposed on the derivatives $\frac{d^k}{dt^k} a(t)$, $k = 0,1,2$ as in earlier work by several authors on this model of nonlinear viscoelastic response.

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A Lower Bound for the Norm of the Solution of a Nonlinear Volterra Equation in One-Dimensional Viscoelasticity.(*)

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1. Introduction and Statement of Results

In a recent paper [1] MacCamy considered the following model for one-dimensional nonlinear viscoelasticity: ($u \equiv$ displacement)

$$(1.1) \qquad u_{tt} = a(0)\sigma(u_x)_x - \int_0^t a_\tau(t-\tau)\sigma(u_x)_x d\tau + \mathfrak{F},$$

on $(0,1)\times[0,\infty)$, subject to initial and boundary data of the form

$$(1.2) \qquad \begin{cases} u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < 1. \end{cases}$$

MacCamy showed that the above initial-boundary value problem has a unique classical solution for all t when $\mathfrak{F}$ is suitably restricted and the data f,g are sufficiently small; furthermore, the solution is asymptotically stable, i e., tends to zero as $t\to+\infty$. The essential hypotheses in [1] are that $a(t) = a_\infty + \mathfrak{a}(t)$, $a_\infty > 0$, $\mathfrak{a} \in L_1(0,\infty)$, $(-1)^k a^{(k)}(t) \geq 0$, $k = 0,1,2$, $\sigma(0) = 0$, $\sigma'(\zeta) \geq \epsilon > 0$, and $|\sigma^{(k)}(\zeta)| \leq \bar{\sigma}$, $k = 0,1,2$ for all ζ, as well as various smoothness assumptions relative to σ, f, g, and $\mathfrak{F}$ and boundedness and growth conditions on $\mathfrak{F}$. In addition, it may be assumed, without any loss of generality, that $a(0) = 1$.

The model (1.1), (1.2) considered by MacCamy is intermediate between two extremes, i.e., the cases where (1.1) is replaced, respectively by the equations

$$(1.3) \qquad u_{tt} = \sigma(u_x)_x, \quad (x,t) \in (0,1)\times[0,\infty)$$

and

$$(1.4) \qquad u_{tt} = \frac{\partial}{\partial x}\left(\sigma(u_x) + \lambda(u_x)u_{xt}\right)$$

Equations (1.3), (1.2) model the standard initial-boundary value problem for one-dimensional nonlinear elasticity and it is well-known that if σ is genuinely nonlinear this problem does not have a smooth global solution for any non-zero data (Lax [9]). Knops [2] has shown that global existence fails for (1.3), (1.2) when there exists a strain-energy function Σ (i.e., $\sigma(\zeta) = \Sigma'(\zeta)$, for all ζ) such that $a\Sigma(\zeta) \geq \zeta\Sigma'(\zeta)$ for some $a > 2$ and all ζ. On the other hand, the initial-boundary value problem associated with (1.4) always has a global smooth solution which is asymptotically stable no matter how large the initial data f,g are ([3], [4]). It is conjectured in [1] that global existence of solutions fails for (1.1), (1.2) if the data are too large.

In the present work we consider the model (1.1) subject to the following assumptions: $a(0) = 1$, $\mathfrak{F} \equiv 0$, and $\sigma(\zeta) = \Sigma'(\zeta)$ with $a\Sigma(\zeta) \geq \zeta\Sigma'(\zeta)$ for all ζ and some $a > 2$. We also assume that there exists $\bar{\sigma} > 0$ (finite) such that $|\sigma'(\zeta)| < \bar{\sigma}$, $\forall \zeta \in R^1$; no sign definiteness conditions are improsed on the $a^{(k)}(t)$, $k = 0,1,2$. Then, under appropriate conditions on the initial data, we will show that any sufficiently smooth solution of (1.1), (1.2) on $[0,T)$, which lies in the class

$$(1.5) \qquad \mathcal{C} = \{u : [0,T) \to H_0^1[0,1] \mid \sup_{[0,T)} \|u\|_{H_0^1} \leq C\},$$

for some real number $C > 0$ must satisfy the quadratic growth estimate

$$(1.6) \qquad \|u\|^2_{L_2} \geq \|f\|^2_{L_2} + 2\nu\|f\|_{L_2} t + \nu^2 t^2 , \quad t \in [0,T) ,$$

where $\nu > 0$ is an appropriately chosen constant. Other growth estimates may also be gleaned from the analysis which we will present below, under various assumptions relative to the initial data, but we do not present these explicitly here.

Remarks. M. Slemrod [8] has recently proved global nonexistence results for a problem which is closely related to (1.1), (1.2). Slemrod considers steady shearing flows in a nonlinear viscoelastic fluid with the shearing stress given by

$$(1.7) \qquad \sigma\left(\int_0^\infty e^{-\lambda\tau} v_x(x,t-\tau)d\tau\right) ,$$

σ a nonlinear, odd, real analytic fucntion, λ a positive constant, and $v(x,t) = \dot{u}(x,t)$ the fluid velocity; using the cited constitutive assumption the evolution equation in [8] has the form

$$(1.8) \qquad u_{tt} = \sigma\left(\int_0^\infty e^{-\lambda\tau} \frac{\partial}{\partial\tau} u_x(x,t-\tau)d\tau\right)_x$$

provided we set the mass density $\rho = 1$. If we assume the same kind of decaying exponential memory in (1.1) as is assumed in [8], allow in (1.1) for a prescribed past history on $(-\infty,0]$, integrate by parts with respect to τ and then make the obvious change of variables in the integral, it is clear that (1.1) is equivalent to

(1.9) $$u_{tt} = -\int_0^\infty e^{-\lambda\tau} \frac{\partial}{\partial\tau} \sigma(u_x(x,t-\tau))_x d\tau$$

The particular choice of history dependence chosen in [8], i.e., $a(t) = e^{-\lambda t}$, and, more essentially, the particular choice of how the nonlinearity $\sigma(\zeta)$ enters the constitutive theory (and, hence the evolution equation (1.8)) allows the boundary-initial history value problem for (1.8) in [8] to be transformed into an initial-boundary value problem for a nonlinear hyperbolic conservation law with linear damping; hyperbolicity for the transformed problem in [8] is shown to be equivalent to the condition $\sigma'(\zeta) > 0$. Once the problem in [8] has been transformed into an equivalent problem for a nonlinear hyperbolic conservation law with linear damping, it is shown to be amenable to analysis based on the use of Riemann invariants and an argument due to Lax [9] to prove that singularities develop, in the solutions of non-linear hyperbolic equations, in finite time for sufficiently large initial data; in fact, Slemrod [8] shows that if the initial data are sufficiently large in certain Sobolev space norms then the second derivatives of the solution develop discontinuities in finite time while the first derivatives remain continuous and bounded(1). The difference in the way in which the nonlinearity σ enters the evolution equations in the models of viscoelastic response considered, respectively, in [1] and [8], i.e., (1.8) as opposed to (1.9), seems to preclude an analysis similar to that in [8] being effected vís a vís the boundary-initial history value problem for (1.9). In fact the transformation in [8] is effected as follows: Clearly (1.8) is equivalent

(1)An exposition of Slemrod's work may be found in his recent Lecture [11] as well as in the forthcoming monograph by this author [13].

$$v_t = \sigma\left(\int_0^\infty e^{-\lambda\tau} v_x(x,t-\tau)\,d\tau\right)_x$$

<u>or</u> $$v_t = \sigma(w)_x$$

<u>if we set</u> $$w(x,t) = \int_0^\infty e^{-\lambda s} v_x(x,t-s)\,ds$$

Now let $r(x,t) = \int_0^\infty e^{-\lambda s} v_t(x,t-s)\,ds$, which is equivalent to

$$r(x,t) = v(x,t) - \lambda \int_0^\infty e^{-\lambda s} v(x,t-s)\,ds$$

and which, in turn, implies that

$$\begin{aligned} r_t(x,t) &= v_t(x,t) - \lambda \int_0^\infty e^{-\lambda s} v_t(x,t-s)\,ds \\ &= v_t(x,t) - \lambda r(x,t) \end{aligned}$$

Clearly, then (1.8) is equivalent to

$$(1.10)\qquad \begin{cases} w_t(x,t) = r_x(x,t) \\ r_t(x,t) = \sigma(w(x,t))_x - \lambda r(x,t) \end{cases}$$

a strictly hyperbolic system if $\sigma'(\zeta) > 0$; a similar reduction is not possible for the evolution equation (1.9)(2). Thus, the global nonexistence results proven by Slemrod in [8] cannot be carried over to the model of one-dimensional nonlinear viscoelastic response considered in [1] so as to deduce global nonexistence of solutions for this case as well. We note that both

(2) Note that the system (1.10) is equivalent to a homogeneous damped nonlinear wave equation for $w(x,t)$, i.e., $w_{tt} + \lambda w_t - \sigma(w)_{xx} = 0$; it is shown in [11] (see also Nohel [12]) that (1.9) is equivalent to a nonhomogeneous damped nonlinear wave equation of the form $u_{tt} + \lambda u_t - \sigma(u_x)_x = \mathscr{S}(x,t)$.

(1.8) and (1.9) reduce to the one-dimensional nonlinear elasticity evolution equation (1.3), for $\lambda = 0$, provided $\lim_{t\to-\infty} |u_x(x,t)| = 0$ uniformly in x, and both constitutive assumptions, which lead to these respective evolution equations, are consistent with the principle of fading memory for simple materials, i.e., [10].

2. Proofs of the Basic Estimates

As a prelude to proving the growth estimate (1.6), delineated in the last section, we first need to derive a suitable expression for the energy associated with our one-dimensional nonlinear viscoelastic model. We define the total energy to be

$$(2.1) \qquad \mathcal{E}(t) \equiv \frac{1}{2}\int_0^1 [\dot{u}(x,t)]^2 dx + \int_0^1 \Sigma(u_x(x,t))dx$$

so that

$$(2.2) \qquad \dot{\mathcal{E}}(t) = \int_0^1 \ddot{u}\dot{u}dx + \int_0^1 \Sigma'(u_x)\dot{u}_x dx$$

$$= \int_0^1 \dot{u}\sigma(u_x)_x dx + \int_0^1 \Sigma'(u_x)\dot{u}_x dx$$

$$- \int_0^1 \dot{u} \int_0^t a_\tau(t-\tau)\sigma(u_x)_x d\tau dt$$

or, in view of the fact that $\Sigma'(\zeta) = \sigma(\zeta)$, for all ζ,

$$(2.3) \qquad \dot{\mathcal{E}}(t) = \int_0^1 \dot{u}\frac{\partial}{\partial x}(\Sigma'(u_x))dx + \int_0^1 \Sigma'(u_x)\dot{u}_x dx$$

$$- \int_0^1 \dot{u} \int_0^t a_\tau(t-\tau) \frac{\partial}{\partial x} (\Sigma'(u_x))d\tau dx$$

$$= \int_0^1 \frac{\partial}{\partial x} (\dot{u}\Sigma'(u_x))dx -$$

$$\int_0^1 \dot{u} \int_0^t a_\tau(t-\tau) \frac{\partial}{\partial x} (\Sigma'(u_x))d\tau dx$$

$$= -\int_0^1 \dot{u}(x,t) \int_0^t a_\tau(t-\tau) \frac{\partial}{\partial x} (\Sigma'(u_x(x,\tau))d\tau dx$$

in view of the boundary conditions (1.2_1). For the sake of convenience we now set $A(t-\tau) = a_\tau(t-\tau)$. Then integration of (2.3) yields

$$\mathcal{E}(t) - \mathcal{E}(0) = -\int_0^t [\int_0^1 \dot{u}(x,\tau) \int_0^\tau A(\tau-\lambda) \frac{\partial}{\partial x} \Sigma'(u_x(x,\lambda))d\lambda dx]d\tau \tag{2.4}$$

However,

$$\frac{d}{d\tau} \int_0^1 u(x,\tau) \int_0^\tau A(\tau-\lambda) \frac{\partial}{\partial x} \Sigma'(u_x(x,\lambda))d\lambda dx =$$

$$\frac{d}{d\tau} \int_0^1 u(x,\tau) \int_0^\tau A(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx =$$

$$\int_0^1 \frac{d}{d\tau} (u(x,\tau) \int_0^\tau A(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda)dx$$

$$= \int_0^1 \dot{u}(x,\tau) \int_0^\tau A(\tau-\lambda)\sigma(u_x(x,\lambda)_x d\lambda dx +$$

$$\int_0^1 u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx +$$

$$A(0) \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx$$

But, $A(0) = A(t-\tau)\Big|_{t=\tau} = \frac{\partial}{\partial\tau} a(t-\tau)\Big|_{t=\tau} = -\dot{a}(0)$

by virtue of our hypotheses on $a(t)$. Thus,

$$\frac{d}{d\tau}\int_0^1 u(x,\tau)\int_0^\tau A(\tau-\lambda)\frac{\partial}{\partial x}\Sigma'(u_x(x,\lambda)d\lambda dx = \tag{2.5}$$

$$\int_0^1 \dot{u}(x,\tau)\int_0^\tau A(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx +$$

$$\int_0^1 u(x,\tau)\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx$$

$$- \dot{a}(0)\int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx$$

Substituting for $\int_0^1 \dot{u}\int_0^\tau A(\tau-\lambda)\sigma(u_x)d\lambda dx$ from (2.5) into (2.4) and performing the indicated integration we obtain

$$\mathcal{E}(t) - \mathcal{E}(0) = -\int_0^1 u(x,t)\int_0^t A(t-\tau)\sigma(u_x(x,\tau))_x d\tau dx \tag{2.6}$$

$$+\int_0^t \left[\int_0^1 u(x,\tau)\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx\right]d\tau$$

$$- \dot{a}(0)\int_0^t\int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau$$

i.e., we have the following

<u>Lemma</u> For the one-dimensional nonlinear viscoelastic model (1.1), (1.2) where $\sigma(\zeta) = \Sigma'(\zeta)$ for all ζ, the total energy $\mathcal{E}(t)$ is given by

$$(2.7) \qquad \mathcal{E}(t) = \frac{1}{2}\int_0^1 g^2(x)dx + \int_0^1 \Sigma(f'(x))dx$$

$$- \int_0^1 u(x,t) \int_0^t A(t-\tau)\sigma(u_x(x,\tau))_x d\tau dx$$

$$+ \int_0^t [\int_0^1 u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx]d\tau$$

$$- a(0) \int_0^t \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau$$

Remarks. We note that $\mathcal{E}(t)$, as defined by (2.1), satisfies $\mathcal{E}(t) \geq 0$, $\forall t \geq 0$; this is a direct consequence of the fact that our assumptions relative to $\sigma(\zeta)$ imply that $\Sigma(\zeta) \geq 0$, $\forall \zeta \in R^1$ (3). In order to see that we must have $\Sigma(\zeta) \geq 0$, $\forall \zeta \in R^1$ we first delineate our basic hypotheses, namely,

$$(2.8a) \qquad \exists \bar{\sigma} > 0 \text{ such that } |\sigma'(\zeta)| \leq \bar{\sigma}, \ \forall \zeta \in R^1$$

$$(2.8b) \qquad \exists \alpha > 2 \text{ such that } \alpha\Sigma(\zeta) \geq \zeta\Sigma'(\zeta), \ \forall \zeta \in R^1$$

Now, by (2.8b), $\Sigma(0) \geq 0$. Also, for $\zeta \neq 0$, $\frac{d}{ds}(\frac{\Sigma(\zeta)}{\zeta^\alpha}) = \frac{\zeta\Sigma'(\zeta) - \alpha\Sigma(\zeta)}{\zeta^{\alpha+1}}$.

Therefore, for $\zeta > 0$, we have by (2.8b), $\frac{d}{d\zeta}(\frac{\Sigma(\zeta)}{\zeta^\alpha}) \leq 0$ so that $\frac{\Sigma(\zeta)}{\zeta^\alpha}$ is nonincreasing on $(0,\infty)$ and $\lim_{\zeta \to 0^+} \frac{\Sigma(\zeta)}{\zeta^\alpha} \geq 0$. By (2.8a) we have that $|\sigma(\zeta)| \leq |\sigma(0)| + \bar{\sigma}|\zeta|$ and $|\Sigma(\zeta)| \leq |\Sigma(0)| + |\sigma(0)|\,|\zeta| + \bar{\sigma}|\zeta|^2$. As $\alpha > 2$,

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(3) This fact was noted by a reviewer who read an earlier version of the manuscript and it is his proof of this fact that we reproduce below.

therefore, we have $\left|\frac{\Sigma(\zeta)}{\zeta^\alpha}\right| \le \frac{|\Sigma(0)| + |\sigma(0)||\zeta| + \bar{\sigma}|\zeta|^2}{\zeta^\alpha} \to 0$ as $\zeta \to +\infty$.

Thus, $\frac{\Sigma(\zeta)}{\zeta^\alpha}$ is nonincreasing on $(0,\infty)$ and satisfies $\lim\limits_{\zeta\to 0^+} \frac{\Sigma(\zeta)}{\zeta^\alpha} \ge 0$ and $\lim\limits_{\zeta\to +\infty} \left|\frac{\Sigma(\zeta)}{\zeta^\alpha}\right| = 0$; hence, $\frac{\Sigma(\zeta)}{\zeta^\alpha} \ge 0$ for $\zeta \ge 0$ which implies that $\Sigma(\zeta) \ge 0$ for $\zeta \ge 0$. Now, define $\Lambda(\zeta) = \Sigma(-\zeta)$. Then $\Lambda'(\zeta) = -\Sigma'(\zeta)$ and, by (2.8b), $\zeta\Lambda'(\zeta) = -\zeta\Sigma'(-\zeta) \le \alpha\Sigma(-\zeta) = \alpha\Lambda(\zeta)$. Repeating the argument given above yields the conclusion that $\Lambda(\zeta) \ge 0$, $\forall\zeta \ge 0$ and, thus, $\Sigma(\zeta) \ge 0$, $\forall\zeta \le 0$. We conclude that $\Sigma(\zeta) \ge 0$, $\forall\zeta \in R^1$, and thu $\mathcal{E}(t) \ge 0$, $\forall t \ge 0$.

We are now ready to prove the main result of this paper, namely, the quadratic growth estimate (1.6) for the L_2 norm of sufficiently smooth solutions of (1.1), (1.2) which lie in the class $\mathcal{C}$. For the remainder of the paper we will assume that $a(t)$ satisfies the smoothness hypotheses

$$\text{(2.8c)} \qquad \sup_{[0,T)} \int_0^t |\dot{a}(t-\tau)|\,d\tau < \infty, \quad \int_0^T \left(\int_0^t \ddot{a}^2(t-\tau)\,d\tau\right)^{1/2} dt < \infty$$

and we set

$$\kappa_T = |\dot{a}(0)|T + \left(1-\frac{1}{\alpha}\right) \sup_{[0,T)} \int_0^t |\dot{a}(t-\tau)|\,d\tau + \sqrt{T} \int_0^T \left(\int_0^t \ddot{a}^2(t-\tau)\,d\tau\right)^{1/2} dt$$

$$\delta = \max\left(\mathcal{E}(0), \bar{\sigma}\kappa_T\mathcal{C}^2\right)$$

Then, we have the following

<u>Theorem</u>. Consider the nonlinear one-dimensional viscoelastic initial-boundary value problem (1.1), (1.2) with $a(0) = 1$ and $\mathfrak{F} = 0$[4]. If

[4] Without loss of generality we may also assume that $\sigma(0) = 0$; for ease of exposition this is assumed in the proof below.

$u \in C^2((0,1)\times[0,T)) \cap \mathcal{C}$ is a solution of (1.1), (1.2) with initial data satisfying

$$(2.9) \qquad \int_0^1 f(x)g(x)dx > \nu\left(\int_0^1 f^2(x)dx\right)^{1/2}, \quad \nu = \frac{2\alpha\delta}{\alpha - 1}$$

then $\|u(t)\|_{L_2(0,1)}$ satisfies the quadratic growth estimate (1.6) on $[0,T)$.

<u>Proof</u>. Define $\mathcal{U}(t) \equiv \|u\|_{L_2}^2 = \int_0^1 u^2(x,t)dx$, $t \geq 0$.

Then

$$(2.10) \qquad \dot{\mathcal{U}}(t) = 2\int_0^1 u\dot{u}dx, \quad \ddot{u}(t) = 2\int_0^1 u\ddot{u}dx + 2\int_0^1 \dot{u}^2 dx$$

or, in view of (1.1)

$$(2.11) \qquad \ddot{\mathcal{U}}(t) = 2\int_0^1 u \frac{\partial}{\partial x} \Sigma'(u_x)dx + 2\int_0^1 \dot{u}^2 dx$$

$$- 2\int_0^1 u \int_0^t A(t-\tau) \frac{\partial}{\partial x} \Sigma'(u_x)d\tau dx$$

$$= 2\int_0^1 \frac{\partial}{\partial x} (u\Sigma'(u_x))dx - 2\int_0^1 u_x\Sigma'(u_x)dx$$

$$- 2\int_0^1 \frac{\partial}{\partial x} \left(\int_0^t A(t-\tau)u(x,t)\Sigma'(u_x(x,\tau)d\tau\right)dx$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

$$+ 2\int_0^1 u^2 dx$$

$$= - 2\int_0^1 u_x\Sigma'(u_x)dx + 2\int_0^1 \dot{u}^2 dx$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

again, in view of the homogeneous boundary data (1.2_1). Adding and subtracting $2\alpha \int_0^1 \Sigma(u_x(x,t))dx$, $\alpha > 2$, in the last line of (2.11) we obtain

$$(2.12) \qquad \ddot{\mathcal{U}}(t) = 2\int_0^1 (\alpha\Sigma(u_x(x,t)) - u_x(x,t)\Sigma'(u_x(x,t))dx$$

$$+ 2\int_0^1 \dot{u}^2(x,t)dx - 2\alpha \int_0^1 \Sigma(u_x(x,t))dx$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

$$\geq 2\int_0^1 \dot{u}^2(x,t)dx - 2\alpha \int_0^1 \Sigma(u_x(x,t))dx$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

in view of our hypothesis (2.8b) relative to $\Sigma(\zeta)$. Combining (2.12_2) with the definition (2.1) of $\mathcal{E}(t)$ now yields

$$(2.13) \qquad \ddot{\mathcal{U}}(t) \geq 2\int_0^1 \dot{u}^2(x,t)dx - 2\alpha(\mathcal{E}(t) - \tfrac{1}{2}\int_0^1 \dot{u}^2(x,t)dx)$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

$$= (2+\alpha) \int_0^1 \dot{u}^2(x,t)dx - 2\alpha\mathcal{E}(t)$$

$$+ 2\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx$$

We now make use of the preceding lemma to substitute for $\mathcal{E}(t)$ in (2.13) and, in this manner, obtain

$$\ddot{\mathcal{U}}(t) \geq (2+\alpha)\int_0^1 \dot{u}^2(x,t)dx - 2\alpha\mathcal{E}(0) \tag{2.14}$$

$$+ 2\alpha\int_0^1 u(x,t)\int_0^t A(t-\tau)\sigma(u_x(x,\tau))_x d\tau dx$$

$$- 2\alpha\int_0^t [\int_0^1 u(x,\tau)\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx]d\tau$$

$$+ 2\alpha\dot{a}(0)\int_0^t\int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau$$

$$+ 2\int_0^1\int_0^t A(t-\tau)u_x(x,t)\Sigma'(u_x(x,\tau))d\tau dx .$$

From (2.10) we have

$$\dot{\mathcal{U}}^2(t) = 4(\int_0^1 u\dot{u}dx)^2 \tag{2.15}$$

and therefore

$$\ddot{\mathcal{U}}(t)\mathcal{U}(t) - (\tfrac{\alpha+2}{4})\dot{\mathcal{U}}^2(t) \geq (2+\alpha)[\int_0^1 u^2 dx \int_0^1 \dot{u}^2 dx \tag{2.16}$$

$$- (\int_0^1 u\dot{u}dx)^2] - 2\alpha\mathcal{U}(t)\mathcal{E}(0)$$

$$- 2\alpha\mathcal{U}(t)\Big(\int_0^t [\int_0^1 u(x,\tau)\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda dx]d\tau$$

$$- \int_0^1\int_0^t A(t-\tau)u(x,t)\sigma(u_x(x,\tau))_x d\tau dx$$

$$- \frac{1}{\alpha}\int_0^1\int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx$$

$$+ \mathring{a}(0) \int_0^t \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau \Bigg)$$

$$\geq 2\alpha\mathcal{U}(t) \Bigg(-(\mathcal{E}(0)) -$$

$$\int_0^1 [\int_0^t u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda d\tau] dx$$

$$+ (1/\alpha) \int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx$$

$$+ \int_0^1 \int_0^t A(t-\tau)u(x,t)\sigma(u_x(x,\tau))_x d\tau dx$$

$$+ \mathring{a}(0) \int_0^t \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau \Bigg)$$

by virtue of the Schwarz inequality and our hypothesis relative to $\mathcal{E}(0)$.

However,

$$+ \int_0^1 \int_0^t A(t-\tau)u(x,t)\sigma(u_x(x,\tau))_x d\tau dx =$$

$$+ \int_0^1 \frac{\partial}{\partial x} (\int_0^t A(t-\tau)u(x,t)\sigma(u_x(x,\tau))d\tau dx$$

$$- \int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx$$

$$= - \int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx$$

in view of the homogeneous boundary conditions relative to $u(x,t)$ and, therefore, (2.16_2) may be rewritten in the form

$$\ddot{\mathcal{U}}(t)\mathcal{U}(t) - \left(\frac{\alpha+2}{4}\right)\dot{\mathcal{U}}^2(t) \geq -2\alpha\mathcal{U}(t)\Big(\mathcal{E}(0) + \int_0^1 \Big[\int_0^t u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda d\tau\Big]dx$$
$$+ \left(1 - \frac{1}{\alpha}\right) \int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx$$
$$- \mathring{a}(0) \int_0^t \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau\Big) \tag{2.17}$$

Our aim, at this point in the proof, is to bound, from above, the three integrals on the right-hand side of (2.17). To this end we have the following series of estimates (beginning with the second integral expression):

$$\left|\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx\right|$$
$$\leq \int_0^t |A(t-\tau)| \int_0^1 |u_x(x,t)||\sigma(u_x(x,\tau))|dx d\tau$$
$$\leq \int_0^t |A(t-\tau)|\left(\int_0^1 u_x^2(x,t)dx\right)^{1/2}\left(\int_0^1 \sigma^2(u_x(x,\tau))dx\right)^{1/2} d\tau$$
$$\leq \|u(\cdot,t)\|_{H_0^1} \int_0^t |A(t-\tau)|\left(\int_0^1 \sigma^2(u_x(x,\tau))dx\right)^{1/2} d\tau$$

However, in view of (2.8a)

$$\int_0^1 \sigma^2(u_x(x,\tau))dx \le \bar{\sigma}^2 \int_0^1 u_x^2(x,\tau)d\tau$$

and, therefore

$$(2.18) \qquad \left|\int_0^1 \int_0^t A(t-\tau)u_x(x,t)\sigma(u_x(x,\tau))d\tau dx\right|$$

$$\le \bar{\sigma}\|u(\cdot,t)\|_{H_0^1} \int_0^t |A(t-\tau)|\,\|u(\cdot,\tau)\|_{H_0^1}\, d\tau$$

$$\le \bar{\sigma}\big(\sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1}\big)^2 \int_0^t |A(t-\tau)|d\tau$$

$$\le \bar{\sigma}\big(\sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1}\big)^2 \sup_{[0,T)} \int_0^t |\mathring{a}(t-\tau)|d\tau$$

Next, if we integrate by parts in the first integral on the right-hand side of (2.17) and estimate as in (2.18) we obtain

$$\left|\int_0^1 \left[\int_0^t u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda d\tau\right]dx\right|$$

$$= \left|\int_0^1 \left[\int_0^t u_x(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))d\lambda d\tau\right]dx\right|$$

$$= \left|\int_0^t \left[\int_0^1 u_x(x,\tau)\left(\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))d\lambda\right)dx\right]d\tau\right|$$

$$\le \int_0^t \left|\int_0^1 u_x(x,\tau)\left(\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda)d\lambda\right)dx\right|d\tau$$

$$\leq \int_0^t \left(\int_0^1 u_x^2(x,\tau)dx\right)^{1/2} \left(\int_0^1 \left(\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))d\lambda\right)^2 dx\right)^{1/2} d\tau$$

$$\leq \sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1} \int_0^t \left(\int_0^1 \left(\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))d\lambda\right)^2 dx\right)^{1/2} d\tau$$

However,

$$\int_0^1 \left(\int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))d\lambda\right)^2 dx$$

$$\leq \int_0^\tau A_\tau^2(\tau-\lambda)d\lambda \int_0^1 \int_0^\tau \sigma^2(u_x(x,\lambda))d\lambda dx$$

$$\leq \left(\int_0^\tau A_\tau^2(\tau-\lambda)d\lambda\right)\bar{\sigma}^2 \int_0^1 \int_0^\tau u_x^2(x,\lambda)d\lambda dx$$

$$= \bar{\sigma}^2 \int_0^\tau \|u(\cdot,\lambda)\|_{H_0^1}^2 d\lambda \int_0^\tau A_\tau^2(\tau-\lambda)d\lambda$$

and, therefore,

$$(2.19) \qquad \left|\int_0^1 \left[\int_0^t u(x,\tau) \int_0^\tau A_\tau(\tau-\lambda)\sigma(u_x(x,\lambda))_x d\lambda d\tau\right]dx\right|$$

$$\leq \bar{\sigma} \sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1} \int_0^t \left(\int_0^\tau A_\tau^2(\tau-\lambda)d\lambda\right)^{1/2} \left(\int_0^\tau \|u(\cdot,\lambda)\|_{H_0^1}^2 d\lambda\right)^{1/2} d\tau$$

$$\leq \bar{\sigma}\sqrt{T} \left(\sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1}\right)^2 \int_0^T \int_0^t A_t^2(t-\tau)d\tau)^{1/2} dt$$

$$= \bar{\sigma}\sqrt{T} \left(\sup_{[0,T)} \|u(\cdot,t)\|_{H_0^2}\right)^2 \int_0^T \left(\int_0^t \ddot{a}^2(t-\tau)d\tau\right)^{1/2} d\tau$$

Finally,

$$(2.20) \qquad \left|\int_0^t \int_0^1 u(x,\tau)\sigma(u_x(x,\tau))_x dx d\tau\right|$$

$$= \left|\int_0^t \int_0^1 u_x(x,\tau)\sigma(u_x(x,\tau)) dx d\tau\right|$$

$$\leq \int_0^t \left(\int_0^1 u_x^2(x,\tau)dx\right)^{1/2} \left(\int_0^1 \sigma^2(u_x(x,\tau))dx\right)^{1/2} d\tau$$

$$\leq \bar{\sigma} \sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1} \int_0^t \left(\int_0^1 u_x^2(x,\tau)dx\right)^{1/2} d\tau$$

$$\leq \bar{\sigma}\, T\left(\sup_{[0,T)} \|u(\cdot,t)\|_{H_0^1}\right)^2$$

Combining the estimates (2.18)-(2.20) with (2.17) then yields the differential inequality

$$(2.21) \qquad \ddot{\mathcal{U}}(t)\mathcal{U}(t) - \left(\frac{\alpha+2}{4}\right)\dot{\mathcal{U}}^2(t)$$

$$\geq -2\alpha\, \mathcal{U}(t)\left(\mathcal{E}(0) + \bar{\sigma}\varkappa_T \left[\sup_{[0,T)} \|\mathcal{U}(\cdot,t)\|_{H_0^1}\right]^2\right)$$

$$\geq -2\alpha\, \mathcal{U}(t)(\mathcal{E}(0) + \bar{\sigma}\varkappa_T C^2)$$

$$\geq -4\alpha\delta\, \mathcal{U}(t)\,,$$

in view of the definitions of $\varkappa_T$, δ and our hypothesis that $u(\cdot,t) \in \mathcal{C}$. If we now set $\gamma = (\alpha-2)/4$ and define $\nu^2 \equiv 2\alpha\delta/(2\gamma+1) = 2\alpha\delta/(\alpha-1)$ then (2.21) has the equivalent form

$$(2.22) \qquad \ddot{\mathcal{U}}(t)\mathcal{U}(t) - (\gamma+1)\dot{\mathcal{U}}^2(t) \geq -2\nu^2(2\gamma+1)\mathcal{U}(t)\,, \quad 0 \leq t < T$$

a differential inequality which has appeared several times in the recent literature on ill-posed initial boundary value problems associated with nonlinear partial differential equations of hyperbolic and parabolic type; indeed, by (2.9) and the definition of $\mathcal{U}(t)$, $\dot{\mathcal{U}}(0) > 0$ and, thus $\dot{\mathcal{U}}(t) > 0$ for $t \in [0,\eta)$. Following the analysis in Levine [14], therefore, we may(5) multiply both sides of (2.22) by $-\gamma(\mathcal{U}^{-\gamma}(t))'\ (\mathcal{U}^{-(\gamma+2)}(t))''$, for $t \in [0,\eta)$ and integrate both sides of the inequality over $[0,t)$ so as to obtain (compare with [14], II-15) for $t \in [0,\eta)$

$$(2.23) \qquad [(\mathcal{U}^{-\gamma}(t)')]^2 - 4\gamma^2\nu^2\mathcal{U}^{-(2\gamma+1)}(t)$$
$$\geq [(\mathcal{U}^{-\gamma}(0))']^2 - 4\gamma^2\nu^2\mathcal{U}^{-(2\gamma+1)}(0) > 0$$

where the right-hand side is positive by virtue of the definition of $\mathcal{U}(t)$ and our hypothesis (2.9) relative to the initial data. Proceeding as in [14] we factor both sides of (2.23) and rewrite the inequality for $t \in [0,\eta]$ as

$$(2.24) \qquad (\mathcal{U}^{-\gamma}(t)' - 2\gamma\nu\mathcal{U}^{-(\gamma+1/2)}(t))(\mathcal{U}^{-\gamma}(t)' + 2\gamma\nu\mathcal{U}^{-(\gamma+1/2)}(t))$$
$$\geq (\mathcal{U}^{-\gamma}(0)' - 2\gamma\nu\mathcal{U}^{-(\gamma+1/2)}(0))(\mathcal{U}^{-\gamma}(0)' + 2\gamma\nu\mathcal{U}^{-(\gamma+1/2)}(0))$$

Again, by (2.9), the factor $\mathcal{U}^{-\gamma}(0)' + 2\gamma\nu\mathcal{U}^{-(\gamma+1/2)}(0)) < 0$ and thus, as neither factor in (2.24) can change sign on $[0,\eta]$ (by virtue of our smoothness assumptions relative to $\mathcal{U}(x,t)$) we have

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(5) At this point we use also the $'$ notation for differentiation w.r.t. t, i.e. $(\ )' = \frac{d}{dt}(\ )$, the new notation being introduced so as to avoid expressions like $\overset{\bullet}{\overline{(\mathcal{U}^{-\gamma}(t))}}$.

$$(2.25)\qquad \mathcal{U}^{-\gamma}(t)' < -2\gamma\nu\,\mathcal{U}^{-(\gamma+1/2)}(t)\,,\quad t\in[0,\eta]$$

Now, suppose that $\dot{\mathcal{U}}(\eta) = 0$. Then by (2.23)

$$\left[[\mathcal{U}^{-\gamma}(t)']^2 - 4\gamma^2\nu^2\mathcal{U}^{-(2\gamma+1)}(t)\right]_{t=\eta}$$

$$= -4\gamma^2\nu^2\mathcal{U}^{-(2\gamma+1)}(\eta) < 0$$

contradicting the fact that (2.23) holds at $t = \eta$. Thus, $\dot{\mathcal{U}}(t) > 0$ for $t\in[0,T)$ and (2.23), as well as (2.25), hold for $t\in[0,T)$; direct integration of (2.25) then yields the estimate

$$(2.26)\qquad \mathcal{U}(t) \geq (\nu t + \mathcal{U}^{1/2}(0))^2\,,\quad 0 \leq t < T$$

which by virtue of the definition of $\mathcal{U}(t)$ is easily seen to be equivalent to (1.6).

QED

Closing Remarks. Dafermos and Nohel [15] have recently generalized the global existence result of MacCamy using energy methods in lieu of Riemann invariants; their method is applicable to problems in more than one space dimension while the Riemann invariant argument is not. Also, the authors in [15] do not require that $|\sigma^{(k)}(\zeta)| \leq \bar{\sigma}$, $k = 0, 1, 2$, $\forall\zeta\in R^1$ but, rather, only that $\sigma\in C^3(R^1)$ with $\sigma(0) = 0$, $\sigma'(0) > 0$. For further details the reader may consult [15] or the extensive discussion of this paper by Dafermos and Nohel which is contained in Bloom [13, chp II]. More recent work on the nonlinear viscoelastic model (1.1), (1.2) includes that of Staffens [16], and Dafermos and Nohel [17].

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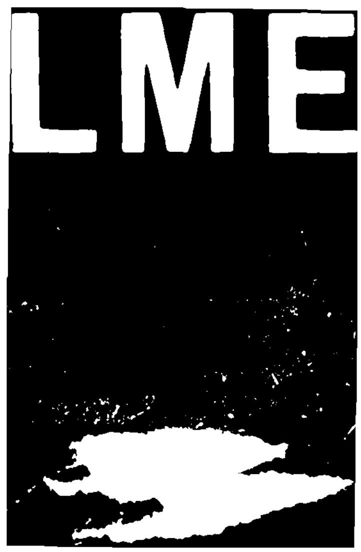
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